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Operator mapping between RNS and extended pure spinor formalisms for superstring

Yuri Aisaka and Yoichi Kazama

Institute of Physics, University of Tokyo
Komaba, Meguro-ku, Tokyo 153-8902 Japan
E-mail: yuri@hep1.c.u-tokyo.ac.jp, kazama@hep1.c.u-tokyo.ac.jp

Abstract: An explicit operator mapping in the form of a similarity transformation is constructed between the RNS formalism and an extension of the pure spinor formalism (to be called EPS formalism) recently proposed by the present authors. Due to the enlarged field space of the EPS formalism, where the pure spinor constraints are removed, the mapping is completely well-defined in contrast to the one given previously by Berkovits in the original pure spinor (PS) formalism. This map provides a direct demonstration of the equivalence of the cohomologies of the RNS and the EPS formalisms and is expected to be useful for better understanding of various properties of the PS and EPS formalisms. Furthermore, the method of construction, which makes systematic use of the nilpotency of the BRST charges, should find a variety of applications.

Keywords: Superstrings and Heterotic Strings, Space-Time Symmetries, BRST Symmetry.

Contents

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1. Introduction

It has already been some time since a new formulation of a superstring in which both the spacetime supersymmetry and the ten-dimensional Poincaré symmetry are manifest has been proposed by Berkovits [1], following earlier attempts [2]–[8]. The central element of this so-called pure spinor (PS) formalism is the BRST-like charge $Q_B = \int [dz]\lambda^\alpha d_\alpha$, where d_α is the spinor covariant derivative and λ^α is a bosonic chiral pure spinor [9]–[11] satisfying the quadratic constraints $\lambda^\alpha \gamma^\mu_{\alpha\beta}\lambda^\beta = 0$. Under these constraints, Q_B is nilpotent and its cohomology was shown to reproduce the physical spectrum of a superstring [12]. All the basic worldsheet fields in this formalism are free and form a centerless conformal field theory (CFT). This allows one to construct Q_B-invariant vertex operators [1, 13] and, together with certain proposed rules, the scattering amplitudes can be computed in a manifestly super-Poincaré covariant manner, which agree with known results [1] and [14]–[16]. Further developments and applications of this formalism are found in [17]–[29], and a comprehensive review, up to a certain point, is available in [30].

Although a number of remarkable features have already been uncovered, many challenges still remain for the PS formalism. The most demanding is the understanding of the underlying fundamental action, its symmetry structures and quantization procedure. In order to achieve this goal, one needs to examine this formalism critically and try to gain as many hints as possible for the proper framework. From this perspective, the non-linear constraints defining the very notion of pure spinor appear to lead to some complications: Not only is it difficult to imagine that a free quantized spinor with constraints emerge naturally in the future fundamental formalism but also the existence of these constraints presents a trouble in defining a proper inner product structure, as pointed out in [29]. Furthermore, as we shall discuss in more detail below, due to the constrained field space one encounters a singular operation in the process of relating the PS formalism to the conventional RNS formalism [16].

Motivated by these considerations, in a recent work [29] we have constructed an extension of the PS formalism, to be referred to as EPS formalism, where the PS constraints are removed. As will be briefly reviewed in the next section, this is achieved by an introduction of a minimum number (five) of fermionic ghost-antighost pairs $(c_{\tilde{I}}, b_I)_{I,\tilde{I}=1\sim 5}$, which properly compensate the effects of the five components of λ^α (and their conjugates) now freed from constraints. It turned out that our formalism fits beautifully into a mathematical scheme known as *homological perturbation theory* [31] and a genuine nilpotent BRST-like charge $\hat{Q}$, the cohomology of which is guaranteed to be equivalent to that of Q_B, was obtained. This scheme also provided a powerful method of constructing the vertex operators, both integrated and unintegrated, which are the extensions of the ones in the PS formalism. Moreover, as an important evidence of the advantage of the extended formalism, we have been able to construct a remarkably simple composite "b-ghost" field $B(z)$, which realizes the fundamental relation $T(z) = \left\{\hat{Q}, B(z)\right\}$, where $T(z)$ is the Virasoro operator of the system. This has never been achieved in the PS formalism.[1] Another advantage of this formalism is that the problem of defining a proper inner product can be solved in the extended space without PS constraints [29].

Besides advantages, we must mention an apparent disadvantage. Namely, due to the extra ghosts $(b_I, c_{\tilde{I}})$, manifest ten-dimensional Lorentz covariance is broken down to U(5) covariance. One need not, however, regard this as a serious problem for two reasons. First, even in the original PS formalism, in order to define the quantized fields properly, one needs to solve the PS constraints and expresses the dependent components of λ^α in terms of independent components. This breaks the manifest symmetry down to U(5). Second, such a breakdown due to the ghosts is expected to be confined in the unphysical sector. Again the situation is very similar to that in the original PS formalism: As argued by Berkovits, Lorentz-noncovariant effects in PS formalism can be decoupled from

[1] Although we do not have a rigorous proof, an analysis presented in section 3.5 strongly indicates that without the extra degrees of freedom introduced in EPS, such a "b-ghost" cannot be constructed.

the physical quantities. Finally, we should mention that an alternative scheme of removing the pure spinor constraints with a finite number of ghosts has been developed in [26]. This formalism has the merit of retaining the Lorentz covariance throughout but to get non-trivial cohomology one must impose an extra condition and this makes the formalism rather involved. Also, another scheme for the superparticle case has been proposed in [27].

One of the important remaining tasks for the EPS formalism is to clarify how one can compute the scattering amplitudes using the vertex operators constructed in [29]. Just as in the PS formalism, one here encounters a difficulty, in particular, concerning the treatment of the zero modes. At the fundamental level, this problem cannot be solved until one finds the underlying action and derives the proper functional measure by studying how to gauge-fix various local symmetries. In the case of PS formalism, Berkovits circumvented this process by ingeniously postulating a set of covariant rules which lead to the known results [1, 14, 15]. Further, the validity of these rules was supported by arguments relating PS to RNS [16]. With the knowledge of the underlying action still lacking, we must resort to similar means. In this paper, we shall construct, as a first step, a precise operator mapping between the EPS and the RNS formalisms entirely in the form of a similarity transformation, which is the most transparent way to connect two theories.

Before explaining our methods and results, we should briefly comment on the corresponding study in the PS formalism [16]. In this work, by making judicious identifications of the fields of the RNS formalism and those of the PS formalism, the extended BRST operator[2] $Q'_{RNS} \equiv \eta_0 + Q_{RNS}$ in the so-called large Hilbert space is expressed in terms of PS variables. Then, certain degrees of freedom of the PS formalism which are missing in the RNS are added in such a way to keep intact the nilpotency of the BRST charge as well as the physical content of the theory. Finally, by a similarity transformation, the BRST charge so modified is mapped to the one appropriate for PS i.e. to Q_B. Although most of these manipulations are rather natural, the similarity transformation employed in the last step contains a singular function and leads to some difficulties, as discussed at some length in [16]. This can be traced to the imposition of the non-linear PS constraints.

Our method to be developed in this paper for connecting the EPS and the RNS formalisms is rather different, and is intimately linked to the scheme of homological perturbation theory. It enables us to construct in a systematic way a complete similarity transformation which maps the BRST-like charge $\hat{Q}$ of the EPS to the extended BRST charge Q'_{RNS} of the RNS, modulo cohomologically trivial orthogonal operators. Due to the absence of the PS constraints, this mapping is entirely well-defined. We believe that this powerful method has not been recognized before and should have many useful applications.

Let us now give an outline of our procedures and results, which at the same time serves to indicate the organization of the paper. After a brief review of the PS and the EPS formalisms in section 2, we begin section 3 with a comparison of the degrees of freedom of EPS, PS and RNS (section 3.1.) This will make it evident that EPS contains extra

[2] η_0 is the zero mode of the η field appearing in the well-known "bosonization" of the β-γ bosonic ghosts [32].

degrees of freedom compared to RNS in the form of two sets of BRST quartets, which we need to decouple. Since this task will be somewhat involved, we shall first consider in section 3.2 a simpler problem of constructing a similarity transformation that connects EPS to PS, in order to illustrate our basic idea. Although such an equivalence was already proven in [29], this provides an alternative more direct proof. After this warm-up, the decoupling of the first quartet is achieved in section 3.3 and that of the second quartet in section 3.4, both by means of similarity transformations. This brings the original $\hat{Q}$ to an extremely simple operator, to be called $\bar{Q}$, plus trivial nilpotent operators which are orthogonal to $\bar{Q}$. Partly as a check of the similarity transformation, we study in section 3.5 how the B-ghost is transformed. This analysis makes more transparent the difficulty of constructing B-ghost in the PS formalism defined in constrained field space. In section 4 we turn our attention to the RNS side and construct, by an analogous method, a similarity transformation which drastically simplifies Q'_{RNS} down to an operator which will be denoted as $\eta_0 + Q_0$. With the BRST charges on both sides reduced to simple forms, it is now an easy matter to establish their relations. In section 5.1, we display the identification of fields given in [16] in appropriate forms, check that these rules produce correct conversion of the energy-momentum tensors and show that in fact the operators $\bar{Q}$ and $\eta_0 + Q_0$ are identical. Finally in section 5.2 we discuss the important problem of the restriction of the proper Hilbert space necessary to achieve the correct cohomology on both sides. This completes the explicit demonstration of the equivalence of EPS, PS, and RNS formalisms. Section 6 is devoted to a brief summary and discussions.

2. A brief review of PS and EPS formalisms

In order to make this article reasonably self-contained and at the same time to explain our notations, let us begin with a very brief review of the essential features of the PS and the EPS formalisms.

2.1 PS Formalism

The central idea of the pure spinor formalism [1] is that the physical states of superstring can be described as the elements of the cohomology of a BRST-like operator Q_B given by[3]

$$Q_B = \int [dz] \lambda^\alpha(z) d_\alpha(z) \,, \tag{2.1}$$

where λ^α is a 16-component bosonic chiral spinor satisfying the pure spinor constraints

$$\lambda^\alpha \gamma^\mu_{\alpha\beta} \lambda^\beta = 0 \,, \tag{2.2}$$

and d_α is the spinor covariant derivative given in our convention[4] by

$$d_\alpha = p_\alpha + i\partial x_\mu (\gamma^\mu \theta)_\alpha + \frac{1}{2}(\gamma^\mu\theta)_\alpha(\theta\gamma_\mu\partial\theta) \,. \tag{2.3}$$

[3]For simplicity we will use the notation $[dz] \equiv dz/(2\pi i)$ throughout.

[4]Our conventions, including normalization, of a number of quantities are slightly different from those often (but not invariably) used by Berkovits.

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x^μ and θ^α are, respectively, the basic bosonic and ferminonic worldsheet fields describing a superstring, which transform under the spacetime supersymmetry with global spinor parameter ϵ^α as $\delta\theta^\alpha = \epsilon^\alpha$, $\delta x^\mu = i\epsilon\gamma^\mu\theta$. x^μ is self-conjugate and satisfies $x^\mu(z)x^\nu(w) = -\eta^{\mu\nu}\ln(z-w)$, while p_α serves as the conjugate to θ^α in the manner $\theta^\alpha(z)p_\beta(w) = \delta^\alpha_\beta(z-w)^{-1}$. θ^α and p_α carry conformal weights 0 and 1 respectively. With such free field operator product expansions (OPE's), d_α satisfies the following OPE with itself,

$$d_\alpha(z)d_\beta(w) = \frac{2i\gamma^\mu_{\alpha\beta}\Pi_\mu(w)}{z-w}\,, \tag{2.4}$$

where Π_μ is the basic superinvariant combination

$$\Pi_\mu = \partial x_\mu - i\theta\gamma_\mu\partial\theta\,. \tag{2.5}$$

Then, due to the pure spinor constraints (2.2), Q_B is easily found to be nilpotent and the constrained cohomology of Q_B can be defined. The basic superinvariants d_α, Π^μ and $\partial\theta^\alpha$ form the closed algebra

$$\begin{aligned}
d_\alpha(z)d_\beta(w) &= \frac{2i\gamma^\mu_{\alpha\beta}\Pi_\mu(w)}{z-w}\,, && (2.6)\\
d_\alpha(z)\Pi^\mu(w) &= \frac{-2i(\gamma^\mu\partial\theta)_\alpha(w)}{z-w}\,, && (2.7)\\
\Pi^\mu(z)\Pi^\nu(w) &= -\frac{\eta^{\mu\nu}}{(z-w)^2}\,, && (2.8)\\
d_\alpha(z)\partial\theta^\beta(w) &= \frac{\delta^\beta_\alpha}{(z-w)^2}\,, && (2.9)
\end{aligned}$$

which has central charges and hence is essentially of *second class.*

Although eventually the rules for computing the scattering amplitudes are formulated in a Lorentz covariant manner, proper quantization of the pure spinor λ can only be performed by solving the PS constraints (2.2), which inevitably breaks covariance in intermediate steps. One convenient scheme is the so-called U(5) formalism,[5] in which a chiral and an anti-chiral spinors λ^α and χ_α, respectively, are decomposed in the following way

$$\begin{aligned}
\lambda^\alpha &= (\lambda_+, \lambda_{IJ}, \lambda_{\tilde{I}}) \sim (1, 10, \overline{5})\,, && (2.10)\\
\chi_\alpha &= (\chi_-, \chi_{\tilde{I}\tilde{J}}, \chi_I) \sim (1, \overline{10}, 5)\,, \qquad (I, J, \tilde{I}, \tilde{J} = 1 \sim 5)\,, && (2.11)
\end{aligned}$$

where we have indicated how they transform under U(5), with a tilde on the $\bar{5}$ indices. On the other hand, a Lorentz vector u^μ is split into $5+\overline{5}$ of U(5) as

$$u^\mu = 2(e_I^{+\mu}u^-_{\tilde{I}} + e_{\tilde{I}}^{-\mu}u^+_I)\,, \tag{2.12}$$

[5]Although our treatment applies equally well to SO(9,1) and SO(10) groups, we shall use the terminology appropriate for SO(10), which contains U(5) as a subgroup. Details of our conventions for U(5) parametrization are found in the [29, Appendix A].

where the projectors $e_I^{\pm\mu}$, defined by $e_I^{\pm\mu} \equiv \frac{1}{2}(\delta_{\mu,2I-1} \pm i\delta_{\mu,2I})$, enjoy the properties

$$
\begin{aligned}
&e_I^{\pm\mu} e_J^{\pm\mu} = 0 \,, \qquad e_I^{\pm\mu} e_J^{\mp\mu} = \frac{1}{2}\delta_{IJ} \,, \\
&e_I^{+\mu} e_{\tilde{I}}^{-\nu} + e_{\tilde{I}}^{-\mu} e_I^{+\nu} = \frac{1}{2}\delta^{\mu\nu} \,.
\end{aligned} \tag{2.13}
$$

In this scheme the pure spinor constraints reduce to 5 independent conditions[6]

$$
\Phi_{\tilde{I}} \equiv \lambda_+ \lambda_{\tilde{I}} - \frac{1}{8}\epsilon_{IJKLM}\lambda_{JK}\lambda_{KL} = 0 \,, \tag{2.14}
$$

and hence $\lambda_{\tilde{I}}$'s are solved in terms of λ_+ and λ_{IJ}. Therefore the number of independent components of a pure spinor is 11 and together with all the other fields (including the conjugates to the independent components of λ) the entire system constitutes a free CFT with vanshing central charge.

The fact that the constrained cohomology of Q_B is in one to one correspondence with the light-cone degrees of freedom of superstring was shown in [12] using the SO(8) parametrization of a pure spinor. Besides being non-covariant, this parametrization contains redundancy and an infinite number of supplimentary ghosts had to be introduced. Nonetheless, subsequently the Lorentz invariance of the cohomology was demonstrated in [19].

The great advantage of this formalism is that one can compute the scattering amplitudes in a manifestly super-Poincaré covariant manner. For the massless modes, the physical unintegrated vertex operator is given by a simple form

$$
U = \lambda^\alpha A_\alpha(x, \theta) \,, \tag{2.15}
$$

where A_α is a spinor superfield satisfying the "on-shell" condition $(\gamma^{\mu_1\mu_2\cdots\mu_5})^{\alpha\beta} D_\alpha A_\beta = 0$ with $D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i(\gamma^\mu\theta)_\alpha \frac{\partial}{\partial x^\mu}$. Then, with the pure spinor constraints, one easily verifies $Q_B U = 0$ and moreover finds that $\delta U = Q_B \Lambda$ represents the gauge transformation of A_α. Its integrated counterpart $\int [dz] V(z)$, needed for calculation of n-point amplitudes with $n \geq 4$, is characterized by $Q_B V = \partial U$ and was constructed to be of the form [1, 2]

$$
V = \partial\theta^\alpha A_\alpha + \Pi^\mu B_\mu + d_\alpha W^\alpha + \frac{1}{2} L^{\mu\nu}_{(\lambda)} F_{\mu\nu} \,. \tag{2.16}
$$

Here, $B_\mu = (i/16)\gamma_\mu^{\alpha\beta} D_\alpha A_\beta$ is the gauge superfield, $W^\alpha = (i/20)(\gamma^\mu)^{\alpha\beta}(D_\beta B_\mu - \partial_\mu A_\beta)$ is the gaugino superfield, $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ is the field strength superfield and $L^{\mu\nu}_{(\lambda)}$ is the Lorentz generator for the pure spinor sector.

With these vertex operators, the scattering amplitude is expressed as $\mathcal{A} = \langle U_1(z_1) U_2(z_2)\; U_3(z_3) \int [dz_4] V_4(z_4) \cdots \int [dz_N] V_N(z_N) \rangle$, and can be computed in a covariant manner with certain rules assumed for the integration over the zero modes of λ^α and θ^α. The proposed prescription enjoys a number of required properties and leads to results which agree with those obtained in the RNS formalism [1] and [14]–[16].

[6]Here and hereafter, for simplicity of notation, we shall denote $\epsilon_{\tilde{I}\tilde{J}\tilde{K}\tilde{L}\tilde{M}}$ as ϵ_{IJKLM}.

2.2 EPS Formalism

Although the PS formalism briefly reviewd above has a number of remarkable features, for the reasons stated in the introduction, it is desirable to remove the PS constraints by extending the field space. Such an extension was achieved in a minimal manner in [29]. Skipping all the details, we give below the essence of the formalism.

Instead of the basic superinvariants forming the essentially second class algebra (2.6)–(2.9), we introduce the four types of composite operators

$$j = \lambda^\alpha d_\alpha \,, \tag{2.17}$$

$$\mathcal{P}_I = \mathcal{N}_I^\mu \Pi_\mu \,, \tag{2.18}$$

$$\mathcal{R}_{IJ} = 2i\lambda_+^{-1}\mathcal{N}_I^\mu(\gamma_\mu\partial\theta)_J \,, \tag{2.19}$$

$$\mathcal{S}_{IJ} = -(\partial\mathcal{N}_I^\mu)\mathcal{N}_J^\mu \,, \tag{2.20}$$

where $\mathcal{N}_I^\mu$ are a set of five Lorentz vectors which are *null*, i.e. $\mathcal{N}_I^\mu\mathcal{N}_J^\mu = 0$, defined by

$$\mathcal{N}_I^\mu \equiv -4(e_I^{+\mu} - \lambda_+^{-1}\lambda_{IJ}e_{\tilde{J}}^{-\mu}) \,. \tag{2.21}$$

Note that j is the BRST-like current of Berkovits *now without PS constraints*. The virtue of this set of operators is that they form a closed algebra which is of *first class*, namely without any central charges. This allows one to build a BRST-like nilpotent charge $\hat{Q}$ associated to this algebra. Introducing five sets of fermionic ghost-anti-ghost pairs $(c_{\tilde{I}}, b_I)$ carrying conformal weights $(0,1)$ with the OPE

$$c_{\tilde{I}}(z)b_J(w) = \frac{\delta_{IJ}}{z-w} \,, \tag{2.22}$$

and making use of the powerful scheme known as homological perturbation theory [31], $\hat{Q}$ is constructed as

$$\hat{Q} = \delta + Q + d_1 + d_2 \,, \tag{2.23}$$

where

$$\delta = -i\int [dz] b_I \Phi_{\tilde{I}} \,, \qquad Q = \int [dz] j \,, \tag{2.24}$$

$$d_1 = \int [dz] c_{\tilde{I}}\mathcal{P}_I \,, \qquad d_2 = -\frac{i}{2}\int [dz] c_{\tilde{I}} c_{\tilde{J}}\mathcal{R}_{IJ} \,. \tag{2.25}$$

The operators (δ, Q, d_1, d_2) carry degrees $(-1, 0, 1, 2)$ under the grading $\deg(c_{\tilde{I}}) = 1$, $\deg(b_I) = -1$, $\deg(\text{rest}) = 0$ and the nilpotency of $\hat{Q}$ follows from the first class algebra mentioned above.

The crucial point of this construction is that by the main theorem of homological perturbation the cohomology of $\hat{Q}$ is guaranteed to be equivalent to that of Q with the constraint $\delta = 0$, i.e. with $\Phi_{\tilde{I}} = 0$, which are nothing but the PS constraints (2.14). Moreover, the underlying logic of this proof can be adapted to construct the massless vertex operators, both unintegrated and integrated, which are the generalization of the ones shown in (2.15) and (2.16) for the PS formalism.

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To conclude this brief review, let us summarize the basic fields of the EPS formalism, their OPE's, the energy-momentum tensor $T_{EPS}(z)$ and the B-ghost field that realizes the important relation $\left\{\hat{Q}, B(z)\right\} = T_{EPS}(z)$. Apart from the $(c_{\tilde{I}}, b_I)$ ghosts given in (2.22), the basic fields are the conjugate pairs $(\theta^\alpha, p_\alpha)$, $(\lambda^\alpha, \omega_\alpha)$, both of which carry conformal weights $(0,1)$, and the string coordinate x^μ. Non-vanishing OPE's among them are

$$\theta^\alpha(z)p_\beta(w) = \frac{\delta^\alpha_\beta}{z-w}\,, \quad \lambda^\alpha(z)\omega_\beta(w) = \frac{\delta^\alpha_\beta}{z-w}\,, \quad x^\mu(z)x^\nu(w) = -\eta^{\mu\nu}\ln(z-w)\,, \tag{2.26}$$

which in U(5) notations read

$$\begin{aligned}
\theta_+(z)p_-(w) &= \frac{-1}{z-w}\,, & \theta_{\tilde{I}}(z)p_J(w) &= \frac{-\delta_{IJ}}{z-w}\,, & \theta_{IJ}(z)p_{\tilde{K}\tilde{L}}(w) &= \frac{\delta^{KL}_{IJ}}{z-w}\,,\\
\lambda_+(z)\omega_-(w) &= \frac{-1}{z-w}\,, & \lambda_{\tilde{I}}(z)\omega_J(w) &= \frac{-\delta_{IJ}}{z-w}\,, & \lambda_{IJ}(z)\omega_{\tilde{K}\tilde{L}}(w) &= \frac{\delta^{KL}_{IJ}}{z-w}\,,\\
x^+_I(z)x^-_{\tilde{J}}(w) &= -\frac{1}{2}\delta_{IJ}\ln(z-w)\,,
\end{aligned} \tag{2.27}$$

where $\delta^{KL}_{IJ} \equiv \delta^K_I\delta^L_J - \delta^K_J\delta^L_I$. The energy-momentum tensor is of the form

$$\begin{aligned}
T_{EPS} &= -\frac{1}{2}\partial x^\mu \partial x_\mu - p_\alpha\partial\theta^\alpha - \omega_\alpha\partial\lambda^\alpha - b_I\partial c_{\tilde{I}} \qquad (2.28)\\
&= -2\partial x^+_I\partial x^-_{\tilde{I}} + p_-\partial\theta_+ + p_I\partial\theta_{\tilde{I}} - \frac{1}{2}p_{\tilde{I}\tilde{J}}\partial\theta_{IJ} + \omega_-\partial\lambda_+ + \omega_I\partial\lambda_{\tilde{I}} - \frac{1}{2}\omega_{\tilde{I}\tilde{J}}\partial\lambda_{IJ} - b_I\partial c_{\tilde{I}}\,,
\end{aligned}$$

with the total central charge vanishing. Finally, the B-ghost field is given by

$$B = -\omega_\alpha\partial\theta^\alpha + \frac{1}{2}b_I\Pi^-_{\tilde{I}}\,. \tag{2.29}$$

3. Similarity transformation in EPS

3.1 Comparison of the field-content of EPS and RNS

As stated in the introduction, our aim in this work is to find a precise operator mapping between the EPS and the RNS formalisms. To do so, we must first compare and clarify the field-content of these formulations. In the U(5) notation, this is given in the following table, where in parentheses contributions to the central charge are indicated:

$$\begin{array}{lcccccccc}
\text{EPS} & x^\pm_I & \begin{pmatrix}\theta_{\tilde{I}}\\ p_I\end{pmatrix} & \begin{pmatrix}\theta_+\\ p_-\end{pmatrix} & \begin{pmatrix}\lambda_+\\ \omega_-\end{pmatrix} & \begin{pmatrix}\lambda_{IJ}\\ \omega_{\tilde{I}\tilde{J}}\end{pmatrix} & \begin{pmatrix}\theta_{IJ}\\ p_{\tilde{I}\tilde{J}}\end{pmatrix} & \begin{pmatrix}\lambda_{\tilde{I}}\\ \omega_I\end{pmatrix} & \begin{pmatrix}c_{\tilde{I}}\\ b_I\end{pmatrix}\\
 & (10) & (-10) & (-2) & (2) & (20) & (-20) & (10) & (-10)\\
\text{RNS} & x^\pm_I & \psi^\pm_I & \begin{pmatrix}b\\ c\end{pmatrix} & \begin{pmatrix}\beta\\ \gamma\end{pmatrix} & & & & \\
 & (10) & (5) & (-26) & (11) & & & &
\end{array} \tag{3.1}$$

On the RNS side, $\psi^\pm_I = e^{\pm\mu}_I\psi_\mu$ are the matter fermions and (b,c) and (β,γ) are the familiar fermionic and bosonic ghosts. By counting the number of bosonic and fermionic fields, one

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sees that, compared to the RNS, the EPS formalism contains extra degrees of freedom forming two "quartets"[7] $(\lambda_{IJ}, \omega_{\tilde{I}\tilde{J}}, \theta_{IJ}, p_{\tilde{I}\tilde{J}})$ and $(\lambda_{\tilde{I}}, \omega_I, c_{\tilde{I}}, b_I)$. (In the case of the original PS formalism, the second quartet is absent.) Therefore it is clear that to connect EPS to RNS, one must decouple these quartets in an appropriate way. This will be done in subsections 3.3 and 3.4.

For the rest of the work, it will be convenient to use the standard "bosonized" representations for the β-γ ghosts [32]. Namely, we write them as

$$\begin{aligned} \beta &= \partial\xi e^{-\phi}\,, \qquad & \gamma &= e^{\phi}\eta\,, \\ \xi &= e^{\chi}\,, \qquad & \eta &= e^{-\chi}\,, \end{aligned} \tag{3.2}$$

where (ξ, η) are fermionic ghosts with dimensions $(0,1)$ and ϕ and χ are chiral bosons satisfying the OPE

$$\phi(z)\phi(w) = -\ln(z-w)\,, \qquad \chi(z)\chi(w) = \ln(z-w)\,. \tag{3.3}$$

Also, for some purposes bosonization of the b-c ghosts as well as the matter fermions $\psi_I^{\pm}$ will be useful as well:

$$c = e^{\sigma}\,, \qquad b = e^{-\sigma}\,, \qquad \sigma(z)\sigma(w) = \ln(z-w)\,, \tag{3.4}$$

$$\psi_I^{+} = \frac{1}{\sqrt{2}}e^{-H_I}\,, \qquad \psi_{\tilde{I}}^{-} = \frac{1}{\sqrt{2}}e^{H_I}\,, \qquad H_I(z)H_J(w) = \delta_{IJ}\ln(z-w)\,. \tag{3.5}$$

The sum of H_I bosons will be denoted by $H \equiv \sum_I H_I$.

Now it is well-known [32] that the RNS string can be formulated either in the small Hilbert space $\mathcal{H}_S$ without ξ_0, i.e. the zero mode of ξ, or in the large Hilbert space $\mathcal{H}_L$ including ξ_0. Since the BRST-like charge $\hat{Q}$ for the EPS formalism contains zero modes of all the relevant fields, one expects that, after the decoupling of the quartets, EPS is connected to the RNS formulated in the large Hilbert space. This will be elaborated further in sections 4 and 5.

3.2 Equivalence of EPS and PS by a similarity transformation

Since the construction of the similarity transformation which decouples the two quartets described above is, as we shall shortly see, somewhat involved, it is instructive to begin with a similar but much simpler task of proving the equivalence of EPS and PS formalisms by the method of similarity transformation, in order to illustrate the basic idea and logic. This equivalence was already proven in our previous paper by the machinary of homological perturbation theory, and hence the following will serve as the second (and more direct) proof.

The goal is to relate the BRST-like charges $\hat{Q}$ and Q_B, for EPS and PS formalisms respectively, by a similarity transformation. With PS constraints imposed, Q_B can be written as

$$Q_B = \int [dz]\hat{\lambda}^{\alpha} d_{\alpha}\,, \tag{3.6}$$

[7] The precise context in which they form quartets will be explained later.

where $\hat{\lambda}^\alpha$ is the pure spinor for which $\lambda_{\tilde{I}}$ components are replaced by $(1/8)\lambda_+^{-1}\epsilon_{IJKLM}\lambda_{JK}\lambda_{LM}$. On the other hand, recalling the form of $\hat{Q}$ given in (2.23)–(2.25), its degree 0 component Q is given by $Q = \int[dz]\lambda^\alpha d_\alpha$ without any constraints on λ^α. Thus, evidently Q and Q_B are related by

$$Q = Q_B + \bar{Q}\,, \qquad \bar{Q} \equiv -\int [dz]\lambda_+^{-1}\Phi_{\tilde{I}} d_I\,. \tag{3.7}$$

To go from $\hat{Q}$ to Q_B, we must obviously remove $\bar{Q}$. To this end, note that $\bar{Q}$ is linear in the PS constraint Φ_I, and hence we should be able to write it as[8]

$$\bar{Q} = \delta R_1\,, \tag{3.8}$$

where R_1 is an integrated operator of degree 1. Such an operator is easily found and is given by

$$R_1 = -i\int [dz]\lambda_+^{-1} c_I d_I\,. \tag{3.9}$$

This suggests that we should use this R_1 as the exponent of the similarity transformation, namely $e^{R_1}\hat{Q}e^{-R_1}$. The relevant calculations are easily performed with the aid of the OPE's between the U(5) components of d_α and Π^μ, which follow from (2.6)–(2.9). After some algebra, we find

$$R_1R_1\delta = 0\,, \qquad R_1 Q = -d_1\,, \qquad R_1 d_1 = -2d_2\,, \qquad R_1 d_2 = 0\,. \tag{3.10}$$

This means that under the similarity transformation each part of $\hat{Q}$ gets transformed as

$$\begin{aligned} e^{R_1}\delta e^{-R_1} &= \delta - \bar{Q}\,, \\ e^{R_1}Q e^{-R_1} &= Q - d_1 + d_2\,, \\ e^{R_1}d_1 e^{-R_1} &= d_1 - 2d_2\,, \\ e^{R_1}d_2 e^{-R_1} &= d_2\,. \end{aligned} \tag{3.11}$$

Adding up, we get a remarkably simple (and expected) result:

$$e^{R_1}\hat{Q}e^{-R_1} = \delta + Q_B\,. \tag{3.12}$$

Since δ and Q_B are nilpotent and anticommute with each other, the main theorem of homological perturbation theory tells us that the cohomology of $\delta + Q_B$ is the same as that of Q_B with δ set to zero, i.e. with the PS constraint $\Phi_{\tilde{I}} = 0$. This proves in a direct way the equivalence of $\hat{Q}$-cohomology and Berkovits' cohomology.

[8]Here and hereafter, a product AB of two integrated operators will always signify the operator product in the sense of conformal field theory and hence is equal to the graded commutator $[A, B\}$. In this notation, the graded Jacobi identity reads $ABC \equiv A(BC) = (AB)C \pm B(AC)$.

3.3 Decoupling of the first quartet

We now launch upon the task of decoupling the quartets by a judicious similarity transformation.

Consider first the decoupling of $(\lambda_{IJ}, \omega_{\tilde{I}\tilde{J}}, \theta_{IJ}, p_{\tilde{I}\tilde{J}})$. To this end, we shall make use of a refined filtration used previously by Berkovits [16]. Namely, we shall assign non-vanishing degrees to the fields we wish to separate in the following way:

$$\mathrm{deg}(p_{\tilde{I}\tilde{J}}) = -2\,, \qquad \mathrm{deg}(\theta_{IJ}) = +2\,, \tag{3.13}$$
$$\mathrm{deg}(\omega_{\tilde{I}\tilde{J}}) = -1\,, \qquad \mathrm{deg}(\lambda_{IJ}) = +1\,, \tag{3.14}$$
$$\mathrm{deg}(\mathrm{rest}) = 0\,. \tag{3.15}$$

Under this grading, $\hat{Q}$ is decomposed into pieces with degrees from -1 up to 6, with degree 4 missing. We have

$$\hat{Q} = \tilde{\delta} + \tilde{Q} + \tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3 + \tilde{d}_5 + \tilde{d}_6\,, \tag{3.16}$$

where $(\tilde{\delta}, \tilde{Q}, \tilde{d}_n)$, which carry degrees $(-1, 0, n)$ respectively, are given by (omitting the integral symbol $\int[dz]$ for simplicity),

$$\begin{aligned}
\tilde{\delta} &= \frac{1}{2}\lambda_{IJ} p_{\tilde{I}\tilde{J}}\,,\\
\tilde{Q} &= -\lambda_+(p_- + 2i\partial x_I^+\theta_{\tilde{I}}) - \lambda_{\tilde{I}}(p_I + 2i\partial x_I^+\theta_+) - ib_I\lambda_+\lambda_{\tilde{I}} - 4c_{\tilde{I}}\partial x_I^+\,,\\
\tilde{d}_1 &= \frac{1}{2}\lambda_{IJ}\left(2i(\theta_{\tilde{I}}\partial x_{\tilde{J}}^- - \theta_{\tilde{J}}\partial x_{\tilde{I}}^-) + (\theta_{\tilde{I}}\partial\theta_{\tilde{J}} - \theta_{\tilde{J}}\partial\theta_{\tilde{I}})\theta_+ - 2\theta_{\tilde{I}}\theta_{\tilde{J}}\partial\theta_+\right) +\\
&\quad + 4c_{\tilde{I}}\lambda_+^{-1}\lambda_{IJ}(\partial x_{\tilde{J}}^- + i(\theta_+\partial\theta_{\tilde{J}} + \theta_{\tilde{J}}\partial\theta_+)) + 4\lambda_+^{-2}c_{\tilde{I}}c_{\tilde{J}}\lambda_{IJ}\partial\theta_+\,, \qquad (3.17)\\
\tilde{d}_2 &= -\lambda_+\theta_{\tilde{I}}(\partial\theta_{IJ}\theta_{\tilde{J}} - \theta_{IJ}\partial\theta_{\tilde{J}}) + \frac{i}{8}\epsilon_{IJKLM}b_I\lambda_{JK}\lambda_{LM} -\\
&\quad -\lambda_{\tilde{I}}(-2i\partial x_{\tilde{J}}^-\theta_{IJ} + \partial\theta_{IJ}\theta_{\tilde{J}}\theta_+ - 2\theta_{IJ}\partial\theta_{\tilde{J}}\theta_+ + \theta_{IJ}\theta_{\tilde{J}}\partial\theta_+) +\\
&\quad + 4ic_{\tilde{I}}(\partial\theta_{IJ}\theta_{\tilde{J}} - \theta_{IJ}\partial\theta_{\tilde{J}}) - 4\lambda_+^{-1}c_{\tilde{I}}c_{\tilde{J}}\partial\theta_{IJ}\,,\\
\tilde{d}_3 &= \frac{i}{2}\epsilon_{IJKLM}\lambda_{IJ}\theta_{KL}\partial x_M^+\,,\\
\tilde{d}_5 &= \frac{1}{4}\lambda_{IJ}\epsilon_{IJKLM}\theta_{KL}(\partial\theta_{MN}\theta_{\tilde{N}} - \theta_{MN}\partial\theta_{\tilde{N}}) + \frac{1}{4}\lambda_{IJ}\epsilon_{JKLMN}\theta_{\tilde{I}}\theta_{KL}\partial\theta_{MN} -\\
&\quad - ic_{\tilde{I}}\lambda_+^{-1}\lambda_{IJ}\epsilon_{JKLMN}\theta_{KL}\partial\theta_{MN}\,, \qquad (3.18)\\
\tilde{d}_6 &= \frac{1}{4}\lambda_{\tilde{I}}\epsilon_{JKLMN}\theta_{IJ}\theta_{KL}\partial\theta_{MN}\,. \qquad (3.19)
\end{aligned}$$

Although these expressions look complicated, what will become important are the relatively simple relations among them which follow straightforwardly from the nilpotency of $\hat{Q}$: Decomposing $\hat{Q}^2 = 0$ with respect to the degree, we have

$$\mathrm{deg} = -2: \qquad \tilde{\delta}^2 = 0\,, \tag{3.20}$$
$$\mathrm{deg} = -1: \qquad \tilde{\delta}\tilde{Q} = 0\,, \tag{3.21}$$
$$\mathrm{deg} = 0: \qquad \frac{1}{2}\tilde{Q}^2 + \tilde{\delta}\tilde{d}_1 = 0\,, \tag{3.22}$$
$$\mathrm{deg} = 1: \qquad \tilde{Q}\tilde{d}_1 + \tilde{\delta}\tilde{d}_2 = 0\,, \tag{3.23}$$

$$
\begin{aligned}
\deg = 2: &\quad \frac{1}{2}\tilde{d}_1^2 + \tilde{Q}\tilde{d}_2 + \tilde{\delta}\tilde{d}_3 = 0\,, && (3.24)\\
\deg = 3: &\quad \tilde{Q}\tilde{d}_3 + \tilde{\delta}\tilde{d}_4 + \tilde{d}_1\tilde{d}_2 = 0\,, && (3.25)\\
\deg = 4: &\quad \tilde{Q}\tilde{d}_4 + \tilde{\delta}\tilde{d}_5 + \frac{1}{2}\tilde{d}_2^2 + \tilde{d}_3\tilde{d}_1 = 0\,, && (3.26)\\
\deg = 5: &\quad \tilde{Q}\tilde{d}_5 + \tilde{\delta}\tilde{d}_6 + \tilde{d}_3\tilde{d}_2 + \tilde{d}_4\tilde{d}_1 = 0\,, && (3.27)\\
\deg = 6: &\quad \tilde{Q}\tilde{d}_6 + \frac{1}{2}\tilde{d}_3^2 + \tilde{d}_4\tilde{d}_2 + \tilde{d}_5\tilde{d}_1 = 0\,. && (3.28)
\end{aligned}
$$

We can now clarify the sense in which the set of fields $(\lambda_{IJ}, \omega_{\tilde{I}\tilde{J}}, \theta_{IJ}, p_{\tilde{I}\tilde{J}})$ form a quartet. From (3.20) and (3.21) we see that $\tilde{\delta}$ is nilpotent and orthogonal to (i.e. anticommutes with) Q. Further, it is trivial to check that $\tilde{\delta}\theta_{IJ} = \lambda_{IJ}$, $\tilde{\delta}\lambda_{IJ} = 0$, $\tilde{\delta}\omega_{\tilde{I}\tilde{J}} = p_{\tilde{I}\tilde{J}}$ and $\tilde{\delta}p_{\tilde{I}\tilde{J}} = 0$. This clearly shows that the above set is a quartet with respect to a BRST-like opeartor $\tilde{\delta}$. Note also that, apart from $\tilde{\delta}$ itself, the members of the quartet appear only in d_n's with positive degrees. Thus, if we can remove these d_n's by a similarity transformation, we will be able to decouple the quartet. This is exactly what we shall achieve below in several steps.

First, consider the nilpotency relation (3.22) at degree 0. It is easy to check that actually each term of this equation vanishes separately, i.e. $\tilde{Q}^2 = 0$ and $\tilde{\delta}\tilde{d}_1 = 0$. The latter relation suggests that $\tilde{d}_1$ may be written as $\tilde{\delta}\tilde{R}_2$ for some degree 2 operator $\tilde{R}_2$. Since by inspection $\tilde{d}_1$ is of the structure $\tilde{d}_1 = \frac{1}{2}\lambda_{IJ}A_{IJ}$ such an operator is readily found:

$$
\begin{aligned}
\tilde{R}_2 = \int [dz]\Big(&-4\,i\lambda_+^{-1}\theta_+ c_{\tilde{I}}\theta_{IJ}\partial\theta_{\tilde{J}} + \theta_+\theta_{\tilde{I}}\theta_{IJ}\partial\theta_{\tilde{J}} + 4\lambda_+^{-2}c_{\tilde{I}}c_{\tilde{J}}\theta_{IJ}\partial\theta_+ + \\
&+ 4\,i\lambda_+^{-1}c_{\tilde{I}}\theta_{\tilde{J}}\theta_{IJ}\partial\theta_+ - \theta_{\tilde{I}}\theta_{\tilde{J}}\theta_{IJ}\partial\theta_+ - 4\lambda_+^{-1}c_{\tilde{I}}\theta_{IJ}\partial x^-_{\tilde{J}} - 2\,i\theta_{\tilde{I}}\theta_{IJ}\partial x^-_{\tilde{J}}\Big)\,.
\end{aligned}
$$

Next, we look at the nilpotency relation (3.23) at degree 1. Substituting $\tilde{d}_1 = \tilde{\delta}\tilde{R}_2$ and using $\tilde{Q}\tilde{\delta} = 0$ and a Jacobi identity, we have $0 = \tilde{Q}\tilde{d}_1 + \tilde{\delta}\tilde{d}_2 = \tilde{Q}(\tilde{\delta}\tilde{R}_2) + \tilde{\delta}\tilde{d}_2 = \tilde{\delta}(\tilde{d}_2 - \tilde{Q}\tilde{R}_2)$. Just as before, this suggests that there exists a degree 3 operator $\tilde{R}_3$ such that $\tilde{d}_2 - \tilde{Q}\tilde{R}_2 = \tilde{\delta}\tilde{R}_3$ holds. After some computation, we find that

$$
\tilde{R}_3 = -\frac{i}{8}\int [dz]\epsilon_{IJKLM}\lambda_{LM}b_I\theta_{JK}\,, \tag{3.29}
$$

satisfies the relation and hence $\tilde{d}_2$ can be written as

$$
\tilde{d}_2 = \tilde{Q}\tilde{R}_2 + \tilde{\delta}\tilde{R}_3\,. \tag{3.30}
$$

Let us go one more step to examine the relation (3.24) at degree 2. Since $\tilde{d}_1^2 = 0$ holds by inspection, using (3.30), the nilpotency of $\tilde{Q}$ and a Jacobi identity, we get $0 = \frac{1}{2}\tilde{d}_1^2 + \tilde{Q}\tilde{d}_2 + \tilde{\delta}\tilde{d}_3 = \tilde{Q}(\tilde{Q}\tilde{R}_2 + \tilde{\delta}\tilde{R}_3) + \tilde{\delta}\tilde{d}_3 = \tilde{\delta}(\tilde{d}_3 - \tilde{Q}\tilde{R}_3)$. By an explicit calculation, one finds that actually a stronger relation $\tilde{d}_3 = \tilde{Q}\tilde{R}_3$ holds.

At this point, one can already see a suggestive structure emerging. Using the expressions for $\tilde{d}_1, \tilde{d}_2$ and $\tilde{d}_3$ obtained so far, $\hat{Q}$ can be rewritten as

$$
\begin{aligned}
\hat{Q} &= \tilde{\delta} + \tilde{Q} + \tilde{\delta}\tilde{R}_2 + \tilde{Q}\tilde{R}_2 + \tilde{\delta}\tilde{R}_3 + \tilde{Q}\tilde{R}_3 \cdots \\
&= \tilde{\delta} + \tilde{Q} - \tilde{R}_2(\tilde{\delta} + \tilde{Q}) - \tilde{R}_3(\tilde{\delta} + \tilde{Q}) + \cdots \\
&= (1 - \tilde{R}_2 - \tilde{R}_3)(\tilde{\delta} + \tilde{Q}) + \cdots\,.
\end{aligned} \tag{3.31}
$$

This is recognized as the beginning of a similarity transformation of the form $\hat{Q} = e^{-(\tilde{R}_2 + \tilde{R}_3 + \cdots)}(\tilde{\delta} + \tilde{Q})e^{\tilde{R}_2 + \tilde{R}_3 + \cdots}$.

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In fact, similar but more involved analysis of the nilpotency relations at higher degrees confirms that this pattern continues to hold and terminates after finite steps, although multiple actions of $\tilde{R}_i$'s occur in non-trivial ways starting at degree 5. Omitting the details, the final answer is given by

$$\begin{aligned}\hat{Q} &= e^{-\tilde{R}}(\tilde{\delta}+\tilde{Q})e^{\tilde{R}}\,,\\ \tilde{R} &= \tilde{R}_2+\tilde{R}_3+\tilde{R}_5+\tilde{R}_6+\tilde{R}_8+\tilde{R}_9\,,\end{aligned} \tag{3.32}$$

where, suppressing $\int[dz]$,

$$\begin{aligned}\tilde{R}_2 = &-4\,i\lambda_+^{-1}\theta_+ c_{\tilde{I}}\theta_{IJ}\partial\theta_{\tilde{J}} + \theta_+\theta_{\tilde{I}}\theta_{IJ}\partial\theta_{\tilde{J}} + 4\lambda_+^{-2}c_{\tilde{I}}c_{\tilde{J}}\theta_{IJ}\partial\theta_+ +\\ &+4i\lambda_+^{-1}c_{\tilde{I}}\theta_{\tilde{J}}\theta_{IJ}\partial\theta_+ - \theta_{\tilde{I}}\theta_{\tilde{J}}\theta_{IJ}\partial\theta_+ - 4\lambda_+^{-1}c_{\tilde{I}}\theta_{IJ}\partial x^-_{\tilde{J}} -\\ &-2\,i\theta_{\tilde{I}}\theta_{IJ}\partial x^-_{\tilde{J}}\,,\end{aligned} \tag{3.33}$$

$$\tilde{R}_3 = -\frac{i}{8}\epsilon_{IJKLM}\lambda_{LM}b_I\theta_{JK}\,, \tag{3.34}$$

$$\begin{aligned}\tilde{R}_5 = &\frac{1}{12}\lambda_+^{-1}\epsilon_{IJKLM}\lambda_{JK}\theta_+\theta_{IN}\theta_{LM}\partial\theta_{\tilde{N}} - \frac{i}{6}\lambda_+^{-2}\epsilon_{IJKLM}\lambda_{JK}c_{\tilde{N}}\theta_{IN}\theta_{LM}\partial\theta_+ +\\ &+\frac{1}{12}\lambda_+^{-1}\epsilon_{IJKLM}\lambda_{JK}\theta_{\tilde{N}}\theta_{IN}\theta_{LM}\partial\theta_+ - \frac{i}{12}\lambda_+^{-1}\epsilon_{IJKLM}\lambda_{JK}\theta_{IN}\theta_{LM}\partial x^-_{\tilde{N}}\,,\end{aligned} \tag{3.35}$$

$$\tilde{R}_6 = \frac{i}{3}\lambda_+^{-1}\epsilon_{IJKLM}c_{\tilde{N}}\theta_{IJ}\theta_{MN}\partial\theta_{KL} - \frac{1}{12}\epsilon_{IJKLM}\theta_{\tilde{N}}\theta_{IJ}\theta_{MN}\partial\theta_{KL}\,, \tag{3.36}$$

$$\tilde{R}_8 = \frac{1}{480}\lambda_+^{-2}\epsilon_{IJKLM}\epsilon_{NPQRS}\lambda_{LM}\lambda_{NS}\theta_{IP}\theta_{JK}\theta_{QR}\partial\theta_+\,, \tag{3.37}$$

$$\tilde{R}_9 = \frac{1}{240}\lambda_+^{-1}\epsilon_{IJKLM}\epsilon_{NPQRS}\lambda_{JM}\theta_{IN}\theta_{KL}\theta_{PQ}\partial\theta_{RS}\,. \tag{3.38}$$

(In deriving this result, one needs some non-trivial identities among several quantities of the type appearing in $\tilde{R}_8$.)

Thus, we have succeeded in reducing our original $\hat{Q}$ to the sum of mutually orthogonal nilpotent operators $\tilde{\delta}$ and $\tilde{Q}$, where the former acts only on the space of the quartet and the latter on the rest of the fields. This shows that the cohomology of $\tilde{\delta}$, which is trivial as already argued, is decoupled and cohomologically $\hat{Q}$ is equivalent to $\tilde{Q}$.

Evidently, our similarity transformations are well-defined for $\lambda_+ \neq 0$, just as in Berkovits' formalism. However, the apparent singularity at $\lambda_+ = 0$ is just a coordinate singularity in the λ-space and does not affect the cohomology, which is known to be Lorentz invariant [19].

3.4 Decoupling of the second quartet

Having reduced $\hat{Q}$ down to $\tilde{Q}$, we now decouple the second quartet $(b_I, c_{\tilde{I}}, \omega_I, \lambda_{\tilde{I}})$ from the cohomology of $\tilde{Q}$. This can be achieved quite analogously as above. Let us assign new non-vanishing degrees to the members of the quartet as follows:

$$\deg(b_I) = -2\,, \qquad \deg(c_{\tilde{I}}) = +2\,, \tag{3.39}$$

$$\deg(\omega_I) = -1\,, \qquad \deg(\lambda_{\tilde{I}}) = +1\,, \tag{3.40}$$

$$\deg(\text{rest}) = 0\,. \tag{3.41}$$

Then, $\tilde{Q}$ is decomposed as

$$\tilde{Q} = \bar{\delta} + \bar{Q} + \bar{d}_1 + \bar{d}_2\,, \tag{3.42}$$

where $(\bar{\delta}, \bar{Q}, \bar{d}_1, \bar{d}_2)$, carrying the degrees $(-1, 0, 1, 2)$, are given by

$$\begin{aligned}
\bar{\delta} &= -i\int [dz]\lambda_+ b_I \lambda_{\tilde{I}}\,, \\
\bar{Q} &= -\int [dz]\lambda_+ \hat{d}_-\,, \\
\bar{d}_1 &= -\int [dz]\lambda_{\tilde{I}}\hat{d}_I\,, \\
\bar{d}_2 &= -4\int [dz] c_{\tilde{I}}\partial x_I^+\,.
\end{aligned} \tag{3.43}$$

In the above, $\hat{d}_-$ and $\hat{d}_I$ are defined as

$$\begin{aligned}
\hat{d}_- &= p_- + 2i\partial x_I^+ \theta_{\tilde{I}}\,, \\
\hat{d}_I &= p_I + 2i\partial x_I^+ \theta_+\,.
\end{aligned} \tag{3.44}$$

Again from the nilpotency $\tilde{Q}^2 = 0$, the relations formally similar to (3.20)–(3.26) follow. Actually these relations reduce in this case to nilpotency of each operator and to simple anticommutation relations among them, except for one non-trivial relation at degree 1 given by

$$\bar{Q}\bar{d}_1 + \bar{\delta}\bar{d}_2 = 0\,. \tag{3.45}$$

Now the relation $\bar{\delta}\bar{d}_1 = 0$ suggests that $\bar{d}_1$ can be expressed as $\bar{d}_1 = \bar{\delta}S_2$, with some operator S_2 of degree 2. It is easily found to be given by

$$S_2 = -i\int [dz]\lambda_+^{-1} c_{\tilde{I}}\hat{d}_I\,. \tag{3.46}$$

Putting this result into (3.45) we get $\bar{\delta}(\bar{d}_2 + S_2\bar{Q}) = 0$. In fact by simple calculations one can check the following properties of S_2:

$$S_2\bar{Q} = -\bar{d}_2\,, \qquad S_2\bar{d}_2 = 0\,, \qquad S_2\bar{d}_1 = 0\,. \tag{3.47}$$

These relations are sufficient to verify the validity of the similarity transformation

$$\tilde{Q} = e^{-S_2}(\bar{\delta} + \bar{Q})e^{S_2}\,. \tag{3.48}$$

Therefore, just as in the previous subsection, the set of fields $(b_I, c_{\tilde{I}}, \omega_I, \lambda_{\tilde{I}})$ form a quartet with respect to the nilpotent operator $\bar{\delta}$ and, as $\bar{\delta}$ and $\bar{Q}$ are mutually orthogonal, they are decoupled from the physical sector governed by the cohomology of $\bar{Q}$. It should also be noted that under this similarity transformation, $\tilde{\delta}$, which played a key role in the previous subsection, is unaffected.

What is rather remarkable is that the information of the non-trivial cohomology in EPS and hence in PS formalism is contained in a drastically simplified nilpotent operator

$$\bar{Q} = -\int [dz]\lambda_+ \hat{d}_- = -\int [dz] \left(\lambda_+ p_- + 2i\lambda_+ \partial x^+_I \theta_{\tilde{I}}\right) . \tag{3.49}$$

In the next section, we shall show that this operator is connected to the BRST charge of the conventional RNS formalism in the large Hilbert space by another similarity transformation.

3.5 Reduction of the B-ghost field

Having decoupled the two quartets and transformed $\hat{Q}$ to a simple operator $\bar{Q}$, it is of interest to see how the B-ghost given in (2.29) gets transformed by the similarity transformation. This analysis will turn out to shed light on the reason why it is difficult to construct its couterpart in the PS formalism formulated in smaller field space.

According to the grading introduced in (3.13)–(3.15), B given in (2.29) is decomposed into the following three pieces with designated degrees:

$$\begin{aligned} B &= B_0 + B_1 + B_4 \,, \\ B_0 &= \frac{1}{2} b_I (\partial x^-_{\tilde{I}} + i(\theta_+ \partial\theta_{\tilde{I}} + \theta_{\tilde{I}}\partial\theta_+)) + \omega_- \partial\theta_+ + \omega_I \partial\theta_{\tilde{I}} \,, \\ B_1 &= -\frac{1}{2}\omega_{\tilde{I}\tilde{J}}\partial\theta_{IJ} \,, \\ B_4 &= -\frac{1}{8}\epsilon_{IJKLM} b_I \theta_{\tilde{J}\tilde{K}} \partial\theta_{\tilde{L}\tilde{M}} \,. \end{aligned} \tag{3.50}$$

It is not difficult to show that under the first similarity transformation $e^{\tilde{R}}(*)e^{-\tilde{R}}$, B is turned into

$$e^{\tilde{R}} B e^{-\tilde{R}} = \tilde{B} \equiv B_0 + B_1 \,. \tag{3.51}$$

In other words, the effect of the similarity transformation is simply to remove the piece B_4.

It should now be noted that by any similarity transformation of the form $e^W(*)e^{-W}$, $W = \int[dz]j(z)$, with $j(z)$ a primary field of dimension 1, the energy momentum tensor $T(w)$ is unchanged. This is because $WT(w) = \int[dz]j(w)/(z-w)^2 = 0$. Due to this property, we must have $(\tilde{\delta} + \tilde{Q})(\tilde{B}_0 + \tilde{B}_1) = T_{EPS}$. Indeed, we find the nice relations

$$\begin{aligned} \tilde{\delta} B_1 &= T_{(p,\theta,\omega,\lambda)} \,, \\ \tilde{Q} B_0 &= T_{EPS} - T_{(p,\theta,\omega,\lambda)} \,, \\ \tilde{\delta} B_0 &= \tilde{Q} B_1 = 0 \,, \end{aligned} \tag{3.52}$$

where $T_{(p,\theta,\omega,\lambda)}$ is the energy-momentum tensor for the first quartet $(\lambda_{IJ}, \omega_{\tilde{I}\tilde{J}}, \theta_{IJ}, p_{\tilde{I}\tilde{J}})$. This shows that B_0 acts as the proper B-ghost in the space without the first quartet, where $\tilde{Q}$ serves as the BRST charge. Next we consider the effect of the second similarity transformation $e^{S_2}(*)e^{-S_2}$. Under the second grading (3.39)–(3.41), B_0 above is split as

$$B_0 = \tilde{B}_{-2} + \tilde{B}_{-1} + \tilde{B}_0 \,, \tag{3.53}$$

where

$$\begin{aligned}\tilde{B}_{-2} &= \frac{1}{2} b_I(\partial x^-_{\tilde{I}} + i(\theta_+ \partial\theta_{\tilde{I}} + \theta_{\tilde{I}}\partial\theta_+))\,,\\ \tilde{B}_{-1} &= \omega_I \partial\theta_{\tilde{I}}\,,\\ \tilde{B}_0 &= \omega_- \partial\theta_+\,. \end{aligned} \tag{3.54}$$

A straightforward computation produces the structure

$$e^{S_2} B_0 e^{-S_2} = \bar{B}_{-2} + \bar{B}_{-1} + \bar{B}_0 + \bar{B}_1\,, \tag{3.55}$$

where $\bar{B}_{-2} = \tilde{B}_{-2}$, $\bar{B}_{-1} = \tilde{B}_{-1}$, $\bar{B}_0 = \tilde{B}_0 + S_2\tilde{B}_{-2}$, and $\bar{B}_1 = S_2\tilde{B}_{-1}$. Since $\tilde{Q}$ is transformed into $\bar{\delta} + \bar{Q}$, we must have $(\bar{\delta} + \bar{Q})(\bar{B}_{-2} + \bar{B}_{-1} + \bar{B}_0 + \bar{B}_1) = T_{EPS} - T_{(p,\theta,\omega,\lambda)}$. In fact, the non-vanishing contributions on the LHS are found to be

$$\begin{aligned}\bar{\delta}\bar{B}_1 &= T_{(b,c,\omega,\lambda)} + \left(b_I c_{\tilde{I}} \partial s + \frac{5}{2}((\partial s)^2 - \partial^2 s)\right)\,,\\ \bar{Q}\bar{B}_0 &= T_{EPS} - T_{(p,\theta,\omega,\lambda)} - T_{(b,c,\omega,\lambda)} - \left(b_I c_{\tilde{I}} \partial s + \frac{5}{2}((\partial s)^2 - \partial^2 s)\right)\,, \end{aligned} \tag{3.56}$$

where $T_{(b,c,\omega,\lambda)}$ is the energy-momentum tensor for the second quartet $(b_I, c_{\tilde{I}}, \omega_I, \lambda_{\tilde{I}})$ and the bosonic field s is defined by $\lambda_+ = e^s$. Therefore, although the sum correctly reproduces $T_{EPS} - T_{(p,\theta,\omega,\lambda)}$, $\bar{B}_0$ cannot be regarded as the B-ghost for the PS formalism. This can be taken as a strong indication that in the constrained field space an appropriate B-ghost field cannot be constructed.

4. Similarity transformation in RNS

4.1 Preliminary

Having decoupled the extra quartets in the EPS, the degrees of freedom now match precisely to the ones in the RNS formulated in the large Hilbert space $\mathcal{H}_L$. As was shown in [16], in $\mathcal{H}_L$ the physical spectrum is characterized by the cohomology of the extended BRST operator[9]

$$Q'_{RNS} = \eta_0 + Q_{RNS}\,, \tag{4.1}$$

where η_0 is the zero mode of η and Q_{RNS} is the usual BRST operator

$$Q_{RNS} = \int [dz] \left[cT_M - \frac{1}{2} e^{\phi}\eta G_M + bc\partial c - c\left(\frac{1}{2}(\partial\phi)^2 + \partial^2\phi + \eta\partial\xi\right) - \frac{1}{4} e^{2\phi} b\eta\partial\eta \right]\,. \tag{4.2}$$

Here T_M and G_M are, respectively, the energy-momentum tensor and the superconformal generator for the matter sector given by

$$\begin{aligned}T_M &= -\frac{1}{2}\partial x^\mu \partial x_\mu - \frac{1}{2}\psi^\mu \partial\psi_\mu\,,\\ G_M &= i\partial x^\mu \psi_\mu\,. \end{aligned} \tag{4.3}$$

[9]This characterization is valid provided that a finite range of "pictures" are used [16]. This point will be elaborated in section 5.2, where we discuss the issue of the correct cohomology.

Our goal in this section is to try to find a similarity transformation which transforms Q'_{RNS} into the simple nilpotent operator $\bar{Q}$, obtained in the previous section, under appropriate identification of fields of EPS and RNS.

Before we begin the consruction, we should mention that in the past an example of a drastic simplification of Q_{RNS} by a similarity transformation has been noted [33]. Namely, it was found that

$$e^W Q_{RNS} e^{-W} = \int [dz] \left(-\frac{1}{4} e^{2\phi} b\eta\partial\eta \right) , \tag{4.4}$$

where

$$\begin{aligned} W &= W_1 + W_2 \,, && (4.5)\\ W_1 &= 2i \int [dz] e^{-\phi} c\xi\psi_\mu \partial x^\mu \,, && (4.6)\\ W_2 &= -2 \int [dz] \partial\phi e^{-2\phi} c\partial c\xi\partial\xi \,. && (4.7) \end{aligned}$$

This remarkable representation found some applications in the context of superstring field theory [34]. At the same time, however, under this transformation η_0 turns into a complicated expression[10]

$$\begin{aligned} \eta_0 + \int [dz] \Big[&-2ie^{-\phi} c\psi_\mu \partial x^\mu + 2e^{-2\phi} c\partial c\xi (\partial x_\mu \partial x^\mu + \psi_\mu \partial\psi^\mu) + \\ &+ e^{-2\phi} \Big(10 c\partial c\xi (\partial\phi)^2 - 8 c\partial c\partial\xi\partial\phi - \frac{10}{3} c\partial^3 c\xi \Big) \Big] \end{aligned} \tag{4.8}$$

so that Q'_{RNS} as a whole is not simplified. Thus, we must seek a different transformation.

4.2 First step

Let us now describe our construction. It will be done in two steps, again by introducing judicious gradings and making use of the relations that follow from the nilpotency of the BRST charge.

As the first step, we adopt the bosonized representation of the β-γ ghosts and assign to the fields the following degrees:

$$\begin{aligned} \deg(\eta, \xi) &= (1, -1) \,, \qquad \deg(c, b) = (5, -5) \,, \\ \deg(\psi_I^+, \psi_I^-) &= (2, -2) \,, \qquad \deg(e^{n\phi}) = n \,, \qquad \deg(\text{rest}) = 0 \,. \end{aligned} \tag{4.9}$$

Then, Q'_{RNS} decomposes into five terms as

$$\begin{aligned} Q'_{RNS} &= \delta + Q_+ + \eta_0 + Q_- + d \,, \\ \delta &\equiv -\frac{1}{4} b\eta\partial\eta e^{2\phi} \,, \\ Q_+ &\equiv -\frac{1}{2} e^\phi \eta G_M^+ \,, \\ Q_- &\equiv -\frac{1}{2} e^\phi \eta G_M^- \,, \\ d &\equiv c \left(T_M - \frac{1}{2} (\partial\phi)^2 - \partial^2\phi - \eta\partial\xi \right) + bc\partial c \,, \end{aligned} \tag{4.10}$$

[10] To our knowledge, this expression has not been recorded in the literature.

where $G_M^\pm$ are defined as

$$\begin{aligned} G_M &= G_M^+ + G_M^- , \\ G_M^+ &= 2i\psi_{\tilde{I}}^- \partial x_I^+ , \qquad G_M^- = 2i\psi_I^+ \partial x_{\tilde{I}}^- . \end{aligned} \tag{4.11}$$

The operators $(\delta, Q_+, \eta_0, Q_-, d)$ carry degrees $(-1, 0, 1, 4, 5)$ respectively. From the nilpotency of Q'_{RNS} and Q_{RNS} we easily find that except for one non-trivial relation

$$Q_+Q_- + \delta d = 0 , \tag{4.12}$$

all the five operators are nilpotent and anticommute with each other. In particular, the relation $\delta Q_+ = 0$ suggests that Q_+ can be written as

$$Q_+ = \delta T , \tag{4.13}$$

with some operator T of degree 1. Such an operator is easily found to be given by

$$T = 2\int [dz] c\xi e^{-\phi} G_M^+ = 4i \int [dz] c\xi e^{-\phi} \psi_{\tilde{I}}^- \partial x_I^+ . \tag{4.14}$$

It is intriguing to note that this operator is precisely "half" of W_1 given in (4.6). For us the importance of this operator is that when acting on η_0 it produces

$$T\eta_0 = Q_0 \equiv 4i \int [dz] c e^{-\phi} \psi_{\tilde{I}}^- \partial x_I^+ , \tag{4.15}$$

which will eventually be identified with the second piece $-2i\lambda_+\theta_{\tilde{I}}\partial x_I^+$ of $\bar{Q}$ in EPS formalism. Moreover, since $TT\eta_0 = 0$, the following similarity transformation holds:

$$e^T \eta_0 e^{-T} = \eta_0 + Q_0 . \tag{4.16}$$

In fact, as we shall later identify η_0 with the first term $-\lambda_+p_-$ of $\bar{Q}$, the RHS of (4.16) will become nothing but $\bar{Q}$ itself. At this point of the analysis, however, it is not yet of great significance since this is only a small part of the similarity transformation and we still have many terms left to be transformed.

Let us study the consequence of the relation (4.12) using the representation (4.13). Since $\delta Q_- = 0$, it can be rewritten as

$$0 = (\delta T)Q_- + \delta d = \delta(TQ_- + d) . \tag{4.17}$$

This suggests that $TQ_- + d$ can be written as

$$TQ_- + d = \delta X , \tag{4.18}$$

for some X of degree 6. By an explicit calculation of the LHS, it is not difficult to show that X is given by

$$X = \int [dz](4\psi_{\tilde{I}}^- \psi_I^+ - 2\partial\phi) e^{-2\phi} c\partial c\xi\partial\xi . \tag{4.19}$$

Again, curiously the second half of this operator is identical to W_2 shown in (4.7).

We are now in a position to look at how the rest of the terms in Q'_{RNS} are transformed under the similarity transformation $e^T(*)e^{-T}$. The commutation relations required for this purpose are easily computed as

$$\begin{aligned} TQ_+ &= 0\,, \qquad TTQ_- = -2Td\,, \qquad TTd = 0\,, \\ TX &= Q_-X = dX = 0\,, \qquad Q_+X = Td\,. \end{aligned} \tag{4.20}$$

They are enough to lead to

$$\begin{aligned} e^TQ_+e^{-T} &= Q_+\,, \\ e^TQ_-e^{-T} &= Q_- - d + \delta X - Td\,, \\ e^T\delta e^{-T} &= \delta - Q_+\,, \\ e^T de^{-T} &= d + Td\,, \end{aligned} \tag{4.21}$$

and, together with the transformation of η_0 already discussed in (4.16), we obtain

$$e^TQ'_{RNS}e^{-T} = \eta_0 + Q_0 + \delta + Q_- + \delta X\,. \tag{4.22}$$

Although we do not display it here, the explicit form of the last term, δX, is rather complicated and it is desirable to remove it before moving on to the next step. This can be achieved by the similarity transformation of the form $e^X(*)e^{-X}$, although it produces an additional term

$$\check{d}_1 \equiv e^X\eta_0e^{-X} = -\int [dz](4\psi^-_{\tilde{I}}\psi^+_I - 2\partial\phi)e^{-2\phi}c\partial c\partial\xi\,. \tag{4.23}$$

In this way, by using the relations given in (4.20) and (4.18), one arrives at

$$e^Xe^TQ'_{RNS}e^{-T}e^{-X} = \check{Q}_{RNS} \equiv \eta_0 + Q_0 + \delta + Q_- + \check{d}_1\,. \tag{4.24}$$

4.3 Second step

Now we proceed to the second step and show that $\check{Q}_{RNS}$ can be brought precisely to the form $\eta_0 + Q_0$ by a further similarity transformation.

To this end, we shall introduce yet another grading scheme and assign to the fields the following degrees:

$$\begin{aligned} \deg(\eta,\xi) &= (-1,1)\,, \qquad \deg(c,b) = (4,-4)\,, \\ \deg\left(e^\phi\right) &= 4\,, \qquad \deg(\text{rest}) = 0\,. \end{aligned} \tag{4.25}$$

Then $\check{Q}_{RNS}$ is decomposed as

$$\check{Q}_{RNS} = \check{\delta} + \check{Q}_0 + \check{d}_1 + \check{d}_2 + \check{d}_3\,, \tag{4.26}$$

where $(\check{\delta}, \check{Q}_0, \check{d}_1, \check{d}_2, \check{d}_3)$ which carry degrees $(-1,0,1,2,3)$ respectively are given by

$$\check{\delta} = \eta_0\,, \tag{4.27}$$

$$\check{Q}_0 = 2\int [dz]ce^{-\phi}G^+_M = Q_0\,, \tag{4.28}$$

$$\check{d}_1 = -\int [dz](4\psi_I^- \psi_I^+ - 2\partial\phi)e^{-2\phi}c\partial c\partial\xi\,, \tag{4.29}$$

$$\check{d}_2 = -\frac{1}{4}\int [dz]b\eta\partial\eta e^{2\phi} = \delta\,, \tag{4.30}$$

$$\check{d}_3 = -\frac{1}{2}\int [dz]e^{\phi}\eta G_M^- = Q_-\,. \tag{4.31}$$

Obviously the new grading merely reorders the previous operators in a convenient way. These operators are all nilpotent and anticommute with each other, except for one non-trivial relation

$$\check{Q}_0\check{d}_3 + \check{d}_1\check{d}_2 = 0\,, \tag{4.32}$$

which follows from $\check{Q}_{RNS}^2 = 0$.

As we wish to remove $\check{d}_2$, let us focus on the relation $\check{Q}_0\check{d}_2 = 0$. By the reasoning repeatedly used, we can find an operator Y_2 of degree 2 such that

$$\check{d}_2 = \check{Q}_0 Y_2\,. \tag{4.33}$$

The explicit form of Y_2 is

$$Y_2 = -\frac{i}{20}\int [dz]e^{3\phi}b\partial b\eta\partial\eta\psi_I^+ x_{\tilde{I}}^-\,. \tag{4.34}$$

Subsitituting (4.33) into (4.32) and rearranging, we immediately find $\check{Q}_0(\check{d}_3 + Y_2\check{d}_1) = 0$. This implies a relation of the form

$$\check{d}_3 + Y_2\check{d}_1 = \check{Q}_0 Y_3 \tag{4.35}$$

for some operator Y_3 of degree 3 and indeed it is given by

$$Y_3 = -\frac{1}{5}\int [dz]e^{2\phi}b\eta(\psi_I^+ x_{\tilde{I}}^-)(\psi_J^+\partial x_{\tilde{J}}^-)\,. \tag{4.36}$$

With this preparation, we now examine a similarity transformation of the form $e^Y(*)\times e^{-Y}$ with $Y = Y_2 + Y_3$. It is straightforward to check the relations

$$Y(\check{d}_2 + \check{d}_3) = 0\,, \qquad Y(\check{Q}_0 + \check{d}_1) = -(\check{d}_2 + \check{d}_3)\,, \qquad Y\check{\delta} = 0\,, \tag{4.37}$$

and from this we easily get

$$e^Y\check{Q}_{RNS}e^{-Y} = \check{\delta} + \check{Q}_0 + \check{d}_1 = \eta_0 + Q_0 + \check{d}_1\,. \tag{4.38}$$

Finally, let us remove $\check{d}_1$. This is done simply by the inverse similarity transformation using the operator X, since $e^{-X}(\eta_0 + \check{d}_1)e^X = \eta_0$ and $XQ_0 = 0$.

Summarizing, after a rather long but systematic procedure, we have established a desired formula

$$e^{-X}e^Ye^Xe^TQ'_{RNS}e^{-T}e^{-X}e^{-Y}e^X = \eta_0 + Q_0\,. \tag{4.39}$$

In the next section, we shall show that the RHS precisely matches the operator $\bar{Q}$ on the EPS side, as promised.

5. Mapping between EPS and RNS

5.1 Identification of fields and BRST operators

To identify the simplified BRST operators on the EPS and the RNS sides, it is necessary to map the basic fields of these formalisms. Fortunately, such a mapping was already proposed by Berkovits in [16] and essentially we only need to make use of this scheme with minor modifications.

Before we give the explicit identification rules, we wish to make a remark. Although a similarity transformation induces redefinition of fields, the identification rules are form-invariant: Both sides of the relations are transformed in the same way so that the OPE's are retained. Thus, we shall find that the conversion rules in [16], which were applied on RNS side *before* various manipulations, remain correct in our case where two theories are connected "in the middle" *after* application of similarity transformations on both sides.

The mapping is best described using the "bosonized" form of various quantities. Besides the ones already described, we introduce, as in [16], a pair of conjugate bosons (s, t) with the OPE

$$s(z)t(w) = \ln(z-w)\,, \qquad s(z)s(w) = t(z)t(w) = 0\,, \tag{5.1}$$

and the energy-momentum tensor

$$T_{st} = \partial s \partial t + \partial^2 s\,. \tag{5.2}$$

Then, λ_+ and its conjugate ω_- can be expressed as

$$\lambda_+ = e^s\,, \qquad \omega_- = \left(\frac{1}{2}\partial s + \partial t\right) e^{-s}\,. \tag{5.3}$$

It is easy to check that the OPE as well as the energy-momentum tensor for (λ_+, ω_-) are correctly reproduced.

The basic mapping can then be described as follows:[11]

$$-e^s p_- = \eta\,, \qquad e^{-s}\theta_+ = \xi\,, \tag{5.4}$$

$$e^t \theta_+ = c\,, \qquad e^{-t} p_- = -b\,, \tag{5.5}$$

$$e^{-s} p_I = -b e^{\phi} \psi_I^+\,, \qquad e^s \theta_{\tilde{I}} = -2c e^{-\phi} \psi_{\tilde{I}}^-\,. \tag{5.6}$$

It is easy to check that they reproduce the correct OPE's on both sides.

A further non-trivial check of the rules above is provided by the correct conversion of the energy-momentum tensors. To demonstrate it, it is convenient to bosonize ξ, η, c, b and $\psi_I^{\pm}$ as in (3.2), (3.4) and (3.5). Then, one can express (s, t) bosons in terms of RNS bosons as [16]

$$s = \sigma - \frac{3}{2}\phi + \frac{1}{2}H\,, \qquad t = -\chi + \frac{3}{2}\phi - \frac{1}{2}H\,. \tag{5.7}$$

[11] Originally, $\theta_+, p_-, \theta_{\tilde{I}}$ and p_I are introduced as independent part of components of the spin fields Σ^α and Σ_α in appropriate pictures [16]. We find it more convenient to display the quantities multiplied by the factors $e^{\pm s}$ in order to avoid the non-trivial Jordan-Wigner factors [35] associated with these spin fields.

Now let us sketch how one can convert the part of T_{EPS} shown in (2.28), with the two quartets dropped, into T_{RNS} by using the correspondence rules above. Since s does not have singular OPE's with η nor ξ, one can easily express p_- and θ_+ in terms of RNS variables using (5.4), and $p_-\partial\theta_+$ can be readily computed. On the other hand, similar manipulations cannot be applied to (5.6) as s does have singular OPE's with the RHS. However, this can be gotten around by combining the finite parts of $(e^{-s}p_I)(z)\partial(e^s\theta_{\tilde{I}})(w)$ and $(e^{-s}p_I)(z)(e^s\theta_{\tilde{I}}\partial s)(w)$, which can be computed easily. In this way one can express $p_I\partial\theta_{\tilde{I}}$ in terms of the RNS bosons. Adding in the RNS expression for T_{st}, many cancellations take place and we indeed reproduce T_{RNS}.

Having established the identification rules, we can easily compare the BRST operators $\bar{Q}$ for EPS and Q_0 for RNS, obtained in the previous section. Recall the form of $\bar{Q}$:

$$\bar{Q} = \int [dz]\left(-\lambda_+ p_- - 2i\lambda_+\theta_{\tilde{I}}\partial x_I^+\right) = \int [dz]\left(-e^s p_- - 2ie^s\theta_{\tilde{I}}\partial x_I^+\right) . \tag{5.8}$$

Applying the map (5.6), we see that this is nothing but $Q_0 = \eta_0 + 4i\int[dz]ce^{-\phi}\psi^-_{\tilde{I}}\partial x_I^+$ and hence the BRST operator for the EPS is directly connected by a series of similarity transformations to the one for the RNS, modulo two quartets which cohomologically decouple.

We wish to emphasize that, in contrast to the corresponding procedure developed by Berkovits for the PS formalism, our transformations do not involve any singular operations or functions. Evidently, this must be due to the use of extended field space, without the PS constraints, in the case of our formalism. As explained in section 3 and 4, construction of our similarity transformations appears very natural, following essentially from the nilpotency structure of the BRST charges.

5.2 Proper Hilbert space and cohomology

Although we have succeeded in connecting the EPS and the RNS formalisms by means of a similarity transformation, there still remains an important question of the proper Hilbert space in which to consider the cohomology.

The generic problem is as follows: Suppose that there exists a local fermionic operator $\Xi(z)$ which is "inverse" to the BRST operator Q in the sense $Q\Xi(z) = 1$. Then any BRST-closed operator $V(z)$ can always be written as a BRST-exact form $V(z) = Q(\Xi(z)V(z))$, since $Q(\Xi(z)V(z)) = (Q\Xi(z))V(z) - \Xi(z)(QV(z)) = V(z)$. Hence in such a situation the cohomology of Q becomes trivial. As was noted by Berkovits [12, 16], such an operator indeed exists in PS formalism and is given (up to an irrelevant overall scale and a Q-exact term) by $\Xi = \lambda_+^{-1}\theta_+$. This operator continues to be the inverse to our BRST operator $\hat{Q}$ in EPS as well. The most natural way to disallow such an operator is, as was postulated by Berkovits [16], to limit the Hilbert space to the so-called ASPC (almost super-Poincaré covariant) subspace. Namely, one allows only those operators which transform covariantly under the spacetime SUSY and U(5) subgroup of the super-Poincaré group. Then since the vertex operators V constructed in (E)PS are known to have ASPC representatives, the products ΞV are not ASPC (due to SUSY-non-invariance of θ_+) and hence are excluded. Evidently the notion of ASPC is still robust upon similarity transformations on the EPS side, although the form of the supercharges get modified. It will be useful to note that the operator $\Xi = \lambda_+^{-1}\theta_+$, on the other hand, can be checked to be form-invariant under these transformations.

Now since the purpose of our work is to relate EPS to RNS, we must also understand how this restriction of the Hilbert space is justified from the point of view of RNS formalism. Let us recall that (E)PS formalism is connected to the RNS formalism in the "large" Hilbert space $\mathcal{H}_l$ with ξ_0 mode, where the extended BRST charge is given by $Q'_{RNS} = \eta_0 + Q_{RNS}$. It was demonstrated in [16] that the cohomology of Q'_{RNS} is equivalent to the conventional cohomology of Q_{RNS} in the "small" Hilbert space without ξ_0, provided that $\mathcal{H}_l$ is restricted to the space of operators with finite range of pictures, to avoid triviality of the cohomology. This implies that if the operator inverse to Q'_{RNS} exists, it must carry infinite range of pictures. Such an operator has not been identified previously, although the inverses to η_0 and Q_{RNS} separately are well-known.

Let us now analyze the nature of the operator Ξ in RNS. According to the correspondence table in section 5.1, this operator is nothing but the familiar ξ ghost in RNS, carrying picture number 1. This is "inverse" to $\bar{Q} = \eta_0 + Q_0$, which is the BRST operator appearing at the juncture of connecting EPS and RNS. It is related to Q'_{RNS} by the similarity transformation on the RNS side:

$$UQ'_{RNS}U^{-1} = \bar{Q}\,, \qquad U \equiv e^{-X}e^{Y}e^{X}e^{T}\,. \tag{5.9}$$

Therefore the counterpart of ξ, to be called $\tilde{\xi}$, in the original RNS formalism is given by

$$\tilde{\xi} = U^{-1}\xi U\,. \tag{5.10}$$

We now note that while X and T carry no picture number the operator Y has picture number 1. Therefore, a similarity transformation by $e^{-Y}(*)e^{Y}$ is capable of producing an operator with infinitely large positive picture.[12] In fact in the case of the operator ξ it is not difficult to prove that the transformed $\tilde{\xi}$ does contain non-vanishing contributions with arbitrarily large picture number.[13] This is as expected of an operator inverse to Q'_{RNS} and such an operator should be excluded from the Hilbert space by the logic of the RNS side.

Thus, we have shown that the cohomology-trivializing operator Ξ can be excluded consistenly in both (E)PS and RNS and this resolves the essential part of the problem. Admittedly, our argument does not show that at the level of the Hilbert space the notion of ASPC in EPS and that of finite picture in RNS are exactly equivalent. In fact it is very unlikely that the image in EPS of the space of RNS operators with finite picture range matches precisely with the space of ASPC operators. However since the crucial operator Ξ is removed in both spaces and the mapping is one to one (modulo decoupling of the quartets), both formulations should have the same non-trivial cohomologies.

[12] It should nevertheless be stressed that this does not always occur; for example, the operator Q'_{RNS} itself is mapped to $\bar{Q}$ with picture number -1. Many other examples like this can be constructed.

[13] Although we shall not give the technical details, the basic reason is that the $e^{3\phi}$ factor present in Y_2 when repeatedly applied produce terms with higher and high pictures together with increasing number of derivatives and product of fermionic fields b, η and ψ_I^+. A systematic counting then shows that the number of derivatives so produced is always sufficient to render such product non-vanishing.

6. Summary and discussions

In this paper, we have succeeded in constructing a similarity transformation which connects, in a well-defined way, the extended version of the pure spinor formalism and the conventional RNS formalism. The BRST charges of these theories are transformed into each other as

$$\begin{aligned} e^{S_2} e^{\tilde{R}} \hat{Q} e^{-\tilde{R}} e^{-S_2} &= \tilde{\delta} + \bar{\delta} + \bar{Q} \,, \\ \bar{Q} &= \eta_0 + Q_0 = e^{-X} e^{Y} e^{X} e^{T} Q'_{RNS} e^{-T} e^{-X} e^{-Y} e^{X} \,, \end{aligned} \tag{6.1}$$

where the operators in the exponent are fully displayed in appropriate sections. We have described the method of construction in some detail since this itself is rather powerful and should find applications in other situations as well. When restricted to the proper Hilbert space discussed in section 5.2, the mapping provides a direct demonstration of the equivalence of the physical spectrum of these two formulations and should prove useful in further investigation of the properties of the EPS and PS formalisms.

One may have an impression that our similarity transformations look rather complicated. Indeed some parts of the calculations required a fair amount of effort, due primarily to the U(5) formalism that had to be used. This feature, however, is very much expected and unavoidable, because highly non-trivial transmutation of SO(9,1) spinors into vectors must inevitably be involved. As is clear from the table summarizing the field-content of EPS and RNS formalisms in section 3.1, only a part of the components of space-time Lorentz spinors in EPS are effective in RNS and this splitting requires U(5) decomposition. In fact space-time spinors in RNS are realized by spin fields, the components of which are not all independent. In this sense, EPS formalism can be regarded as realizing a linearization of the spinor representation in a larger field space. Deeper understanding of such connections would require a discovery of a universal fundamental action from which one can derive EPS and RNS formalisms.

Even with such an underlying action still lacking, by appropriately mapping the firm knowledge available for RNS to the EPS side, one should be able to gain deeper understanding of the properties of the EPS and PS formalisms, for example the origin of the rules of computation of the scattering amplitudes, how to handle loops, etc. Such a work is underway and we hope to report our findings in a future communication.

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